Harmonicity in Networked Social Information Dynamics

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Abstract

A new class of mathematical models for the structure and evolution of information over social networks is proposed, inspired by ideas from algebraic topology. These models have the advantage of being strongly heterogeneous, including differentiating between individuals' internal states and their pairwise external expressions thereof. As well, these models admit a variety of data types as target representations: vector valued, lattice-valued, and more, all possessing a common framework. Finally, these models admit universal notions of diffusion and harmonic state distributions.

On Mathematical Models of Networked Social Information

The dynamics of social information – opinions, preferences, propaganda, and more – have sparked a vast set of models in the intersection of the mathematical and social sciences.

Key classic models for social information dynamics have common features (Taylor 1968, DeGroot 1974). Influence is subordinate to a social network, modeled as a (directed or undirected) graph. The *graph Laplacian* is used as a diffusion operator to evolve initial states to consensus over the network. Such consensus results, unrealistic in most settings, prompt generalizations to a multitude of models that admit polarization and related features as solutions. Many models introduce heterogeneity by inserting malicious or stubborn agents, selective falsification, bounded confidence, inhibiting influence, susceptibility, and much more.

Two Pushouts

Of the many existing and possible generalizations of social information dynamics models, there are two axes along which one can push with respect to heterogeneity.

Internal State Spaces

Typically, models assume that agents have a state, be it an opinion, preference, or degree of acceptance of a particular piece of information. Scalar models encode each agent's real-valued opinion on some fixed topic; these have been generalized to vector-valued models recording opinions on multiple topics. Discrete-state models tend to be binary (e.g., infected or not) or other sets with a total order (e.g., number of stars in a ratings system).

The use of scalar or ordered values is more for the convenience of modelling than fidelity to how agents internally process opinions or preferences. A private, internal ideology as a framework from which opinions or preferences emanate is, one suspects, both an intricate and hidden structure.

One candidate for a state space which has received little attention is a *lattice*, a partially-ordered set with algebraic operations of *join* and *meet* (generalizations of *max* and *min*). While partially-ordered sets have seen use in modelling complex preference relations (Janis et al. 2015), lattices have the additional expressive power of subsuming Boolean algebras and other logics. The ability to model internal state spaces as personalized logics is particularly appealing.

Internal – Expression – External

In the classical models, each agent is influenced by neighbors sharing their states. This raises several issues. If each agent has a personal state space, how are these compared to determine influence or consensus? Does communication require full disclosure of internal hierarchies? Avoiding such questions is another benefit of classical simple models.

One is led to the conception of heterogeneous internal state spaces assigned to agents – to vertices in the network graph. To communicate with a neighbor, one wants an external representation, something like a state space, but assigned to the edge between vertices in the network. This discourse space may be completely distinct from the internal state spaces of either party. For example, one can discuss opinions on a matter for which there are no internal priors: one formulates an opinion based on internal ideology.

It is the formulation of the public expression based on the private state which is an interesting aspect of influence and interaction over a social network – herein lies obfuscation, modulation, and lying.

Sheaves for Social Information

There is a simple structure which bundles together the internal states and the external expressions. One of the most powerful branches of mathematics intended specifically for dealing with inhomogeneous structures is *sheaf theory*. Developed in the mid 20th century, sheaf theory is a core tool in algebraic geometry and topology.

Though the general theory is rarified (Bredon 1997), there is a simpler variant adapted to networks that has of late been useful in a variety of settings, including, most recently, opinion dynamics. Such a network sheaf is a data structure, $\mathcal{F}$, over an undirected network of vertices and edges; $\mathcal{F}$ assigns to each vertex and each edge an algebraic object, such as a vector space, all tied together with actions on the data – transformations from vertex data to edge data. The sheaf is heterogeneous: each vertex and edge has its own space of data, perhaps of different dimension and/or type.

For example (Hansen-Ghrist 2021), a *discourse sheaf* $\mathcal{F}$ over a network has vertex data given by a vector space of *basis opinions*: each vector records real-valued opinions with respect to a basis of topics. Such opinion spaces are private and differ in dimension from vertex to vertex. Over each edge the sheaf assigns a vector space of expressed opinions over another fixed basis of discussion topics: a *discourse space*. The sheaf is outfitted with linear transformations from opinion spaces to discourse spaces. These express public opinions based on private ideologies.

Thanks to the strongly heterogeneous nature of a sheaf, agents do not need to openly express private opinions or preferences. Indeed, a sheaf model permits exaggeration or outright falsification, selectively, neighbor-by-neighbor.

Sheaves can take values in categories beyond vectors of real numbers; sheaves of lattices over a network have recently been investigated (Ghrist-Riess 2022).

Dynamics & Diffusion

Network models with greater fidelity still require the ability to evolve internal states and external expressions over time. In the context of network sheaf models there is a natural universal construction which permits diffusion.

Sheaf Laplacians

The classic graph Laplacian – at the foundation of so many network dynamics models – has a generalization to sheaves, with explicit formulae for (1) sheaves of based vector spaces (Hansen-Ghrist 2021), (2) sheaves of lattices (Ghrist-Riess 2022), and more. In its simplest manifestation, the sheaf Laplacian is an operator which acts on a distribution of vertex data, updating it by pairwise communication.

A qualitative description is apt. Assume a choice of internal state for every agent. An agent reads the expressed public opinions/preferences of each neighbor; then computes "*What would I have to change to express agreement with them?*"; then, these changes are merged over all communications with neighbors. Note that this operation maintains private, internal states: only external expressions are read.

A distribution of internal states is said to be *harmonic* if the action of the Laplacian is null; that is, each agent has an internal state which when expressed to its neighbors signals agreement. It is not a strict consensus – what does it mean for two agents with entirely different internal ideologies to agree? – but rather an expressed external consensus: a truce.

Harmonic Convergence

The sheaf Laplacian induces dynamics that drive a network toward a harmonic state. A heat equation (in continuous or discrete time) using the sheaf Laplacian evolves any initial internal state distribution to the closest harmonic distribution. This is a simple model where everyone communicates with neighbors and adjusts their internal state slightly to reduce the aggregate disagreement with neighbors.

More complex dynamical models are worth attention. It is here that the framework of harmonic geometry is genuinely useful. Initial explorations show that stubborn agents (propagandists) in a diffusion model drive a system to the harmonic extension over the propaganda subset (Hansen-Ghrist 2021). Other features (weights, bounded confidence, etc.) are likewise incorporable. The models hinted at here are deterministic; adapting stochastic models to this framework remains a compelling open problem.

References

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